$$=\frac{4}{3a}\int_{x_2}^{x_1}\frac{x(2a^2+ax)^{\frac{3}{2}}}{(a+x)\sqrt{\{4ac^2(a+x)-x^2\}}}dx$$

$$=\frac{4a}{3}\int_{x_2}^{x_1}\frac{x\sqrt{\{2a^2+ax\}}dx}{(a+x)\sqrt{\{4ac^2(a+x)-x^2\}}}+\tfrac{4}{3}\int_{x_2}^{x_1}\frac{x\sqrt{\{2a^2+ax\}}dx}{\sqrt{\{4ac^2(a+x)-x^2\}}}.$$

Let $x=2ac^2+2ac\sqrt{\{1+c^2\}}\cos 2\theta$, $1+c^2+c\sqrt{\{1+c^2\}}=b^2$, $2c\sqrt{\{1+c^2\}}/b^2=e^2$, $4c\sqrt{\{1+c^2\}}/\{2b^2-1\}=d$.

$$\therefore\ S_p=\frac{16a^2b\sqrt{2(b^2-1)}}{3(2b^2-1)}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-d\sin^2\theta)\sqrt{\{1-e^2\sin^2\theta\}}}$$

$$-\frac{16a^2b\sqrt{2(2c\sqrt{\{1+c^2\}}+b^2e^2-e^2)}}{3(2b^2-1)}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{(1-d\sin^2\theta)\sqrt{\{1-e^2\sin^2\theta\}}}$$

$$+\frac{32a^2bce^2\sqrt{\{2(1+c^2)\}}}{3(2b^2-1)}\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{(1-d\sin^2\theta)\sqrt{\{1-e^2\sin^2\theta\}}}$$

$$+\tfrac{16}{3}a^2b\sqrt{2(b^2-1)}\int_0^{\frac{1}{2}\pi}\sqrt{\{1-e^2\sin^2\theta\}}d\theta$$

$$-\tfrac{32}{3}a^2bc\sqrt{\{2(1+c^2)\}}\int_0^{\frac{1}{2}\pi}\sqrt{\{1-e^2\sin^2\theta\}}\sin^2\theta d\theta.$$

$$\therefore\ S_p=A\Pi(e,\ -d,\ \tfrac{1}{2}\pi)+\frac{B}{d}\{F(e,\ \tfrac{1}{2}\pi)-\Pi(e,\ -d,\ \tfrac{1}{2}\pi)\}$$

$$+\frac{C}{d^2e^2}\{e^2\Pi(e,\ -d,\ \tfrac{1}{2}\pi)+dE(e,\ \tfrac{1}{2}\pi)-(d+e^2)F(e,\ \tfrac{1}{2}\pi)\}+DE(e,\ \tfrac{1}{2}\pi)$$

$$+\frac{E}{3e^2}\{1-2e^2)E(e,\ \tfrac{1}{2}\pi)-(1-e^2)F(e,\ \tfrac{1}{2}\pi)\}.$$

$$\therefore\ S=4\pi a^2c(1+c^2)+(A-\frac{B}{d}+\frac{C}{d^2})\Pi(e,\ -d,\ \tfrac{1}{2}\pi)+\Big(\frac{C}{de^2}+D$$

$$+\frac{E(1-2e^2)}{3e^2}\Big)E(e,\ \tfrac{1}{2}\pi)+\Big(\frac{B}{d}-\frac{C(d+e^2)}{d^2e^2}-\frac{E(1-e^2)}{3e^2}\Big)F(e,\ \tfrac{1}{2}\pi).$$

MECHANICS.

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139. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College. Defiance, Ohio.**

A homogeneous sphere, radius $r=50$ inches, makes $m=30$ revolutions around an axis every second. The mass begins to disappear from the surface into space at a rate exactly sufficient to cause the diameter to decrease uniformly at the rate of $(1/n)$th$=1/1000$th of a linear inch per second. At what rate per second is the angular velocity of the sphere changing the instant the diameter becomes $p=10$ inches? What is the diameter of the sphere when the rate of disappearance of matter is midway between minimum and maximum? When is the angular velocity a maximum? How does this maximum angular velocity compare with the original angular velocity? What is the diameter of the sphere when the paracentric force is (1) a maximum, and (2) a minimum?

No solution of this problem has been received.

140. Proposed by J. F. LAWRENCE, A. B., St. Louis. Mo.

A long row of particles, each mass m, is placed on a smooth horizontal table. Each is connected with the two adjacent ones by similar light elastic strings of natural length l. They receive small longitudinal disturbances such that each of them proceeds to perform a harmonic oscillation. Prove that there will be two waves of vibration in opposite directions with the same velocity, viz, $l'\sqrt{\frac{E}{ml}}\frac{q}{\pi}\sin\frac{\pi}{q}$, when l' is the average distance between two successive particles, q the number of intervals between two particles in the same phase, and E the modulus of elasticity. [*Mathematical Tripos*, 1873.]

Solution by G. B. M ZERR. A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia. Pa.

Let $D=d/dt$, and the equation of motion is $y_{k+1}-2y_k+y_{k-1}=\frac{D^2}{c^2}y_k$ where $c^2=E/(ml)$. To solve this equation of differences, treat D as constant and put $y_k=Cx^k$ where C and x are two constants. Substituting and reducing, $x-2+1/x=(D/c)^2$.

$$\therefore \sqrt{x}-1/\sqrt{x}=\pm D/c. \quad \sqrt{x}+1/\sqrt{x}=\pm\sqrt{\{4+(D/c)^2\}}=\pm 2\sqrt{1+\left(\frac{D}{2c}\right)^2}$$

$$\therefore \sqrt{x}=\sqrt{1+\left(\frac{D}{2c}\right)^2}-\frac{D}{2c}=E.$$

$$\therefore y_k=E^{2k}f(t)+E^{-2k}F(t).$$

Let us express $f(t)$ and $F(t)$ in a series whose general term is $A\cos(2c\sin\theta t+\omega)$.

The operator E under the radical contains only even powers of D and we can write $-(2c\sin\theta)^2$ for D^2.

$$\therefore E\cos(2c\sin\theta t+\omega)=\cos(2c\sin\theta t+\omega-\theta).$$

$$\therefore E^{2k}\cos(2c\sin\theta t+\omega)=\cos(2c\sin\theta t+\omega-2k\theta).$$

$$E^{-2k}\cos(2c\sin\theta t+\omega)=\cos(2c\sin\theta t+\omega+2k\theta).$$

$$\therefore y_k=\Sigma A\cos(2c\sin\theta t+\omega-2k\theta)+\Sigma B\cos(2c\sin\theta t+\omega+2k\theta).$$

If we substitute in any one term of the first series $k+1$ for k and $t+T$ for t, where $T=\frac{\theta}{c\sin\theta}$, the term is unaltered..